§ 3.3 Cauchy criterion

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

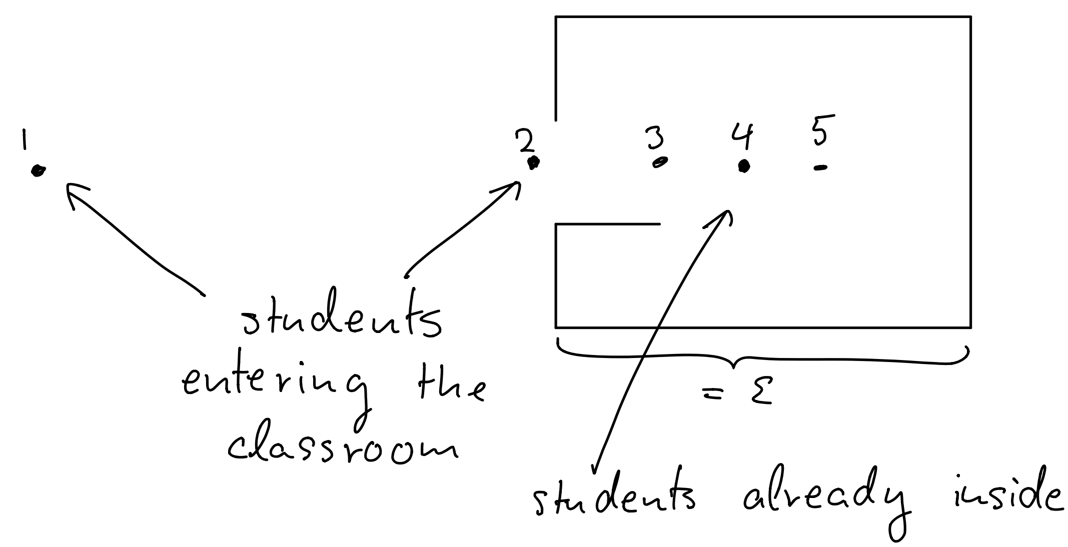
Definition 3.2:

$(a_n)_{n\in\mathbb{N}}$ is a "Cauchy sequence", if

$$\forall \ \varepsilon > 0 \ \exists \ n_0 = n_0(\varepsilon) \in \mathbb{N} \ \forall \ n, \ell \geq n_0 : |a_n - a_\ell| < \varepsilon$$

Example 3.3:

Take a classroom of size ε



Then the sequence of students entering this classroom is a cauchy sequence.
(distance between students already inside is less than ε)

Proposition 3.4:

Convergent sequences are Cauchy sequences

Proof:

Let $a_n \longrightarrow a \ (n \to \infty)$. For $\varepsilon > 0$ choose $n_0 = n_0(\varepsilon)$ s.t.

$$\forall \ n \geq n_0 : |a_n - a| < \varepsilon$$

Then we have

$$\forall \ l, n \geq n_0 : \ |a_n - a_l| \leq |a_n - a| + |a - a_l| < 2\varepsilon$$

$\square$

Proposition 3.5:

Cauchy sequences in $\mathbb{R}$ are convergent.

Example 3.4:

Set $a_1 = 1$, $a_n = a_{n-1} + \frac{1}{n}$, $n \geq 2$

Then we obtain the "harmonic sequence":

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}$$

The sequence $(a_n)_{n \in \mathbb{N}}$ is divergent as for all $n \in \mathbb{N}$ we have

$$a_{2n} - a_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}$$

$\Rightarrow (a_n)_{n\in\mathbb{N}}$ is <u>not</u> a Cauchy sequence and diverges according to Prop. 3.4.

§3.4 Supremum and Infimum

Definition 3.3

A set $A \subset \mathbb{R}$ is "bounded from above", if there exists a number $b \in \mathbb{R}$ s.t.,

$$\forall\, a \in A : \quad a \leq b$$

Every such b is an "upper bound" for A.

Analogously, one defines "bounded from below" and "lower bound".

Example 3.5:

i) In our Calculus class the age of every student is bounded from above by 100 years.

ii) The interval

$$(-1,1) = \{x \in \mathbb{R} \mid -1 < x < 1\}$$

is bounded from above (by $b=1$) and from below (e.g. by $a=-1$).

iii) $\mathbb{N} = \{1, 2, 3, \dots\}$ is bounded from below, but unbounded from above.

Let now $\emptyset \neq A \subset \mathbb{R}$ be bounded from above.

Then we have

$$B = \{ b \in \mathbb{R} \mid b \text{ is an upper bound for } A \} \neq \emptyset$$

and $\forall\ a \in A,\ b \in B:\ a \leq b$

The completeness axiom then gives the existence of a number $c \in \mathbb{R}$ s.t.

$$\forall\ a \in A,\ b \in B:\ a \leq c \leq b. \qquad (*)$$

<u>Remark 3.2:</u>

Aparently, c is an upper bound for a, so $c \in B$. As $c \leq b$ for all $b \in B$ at the same time, c is the "smallest upper bound" for A, and c is uniquely determined by $(*)$.

<u>Definition 3.4:</u>

For a set $A \subset \mathbb{R}$ which bounded from above, the number $c := \sup A$ defined by $(*)$ is called the "supremum" of A.

We summarize:

<u>Proposition 3.6:</u>

i) Every set $\emptyset \neq A \subset \mathbb{R}$ bounded from above has a smallest upper bound $c = \sup A$.

ii) Analogously, every from below bounded set $\emptyset \neq A \subset \mathbb{R}$ admits a biggest lower bound $d = \inf A$, the so called "infinum" of A.

<u>Example 3.6:</u>

i) The infinum of the age of all humans on earth is 0 years.

ii) Let $A = (-1, 1) \subset \mathbb{R}$. Then we have

$$\sup A = 1, \quad \inf A = -1$$

iii) $\mathbb{N} = \{1, 2, \dots\}$ is unbounded from above, thus does not admit a supremum.
On the other hand, every $k \in \mathbb{N}$ satisfies $k \geq 1$. As $1 \in \mathbb{N}$, 1 is the biggest lower bound, that is,

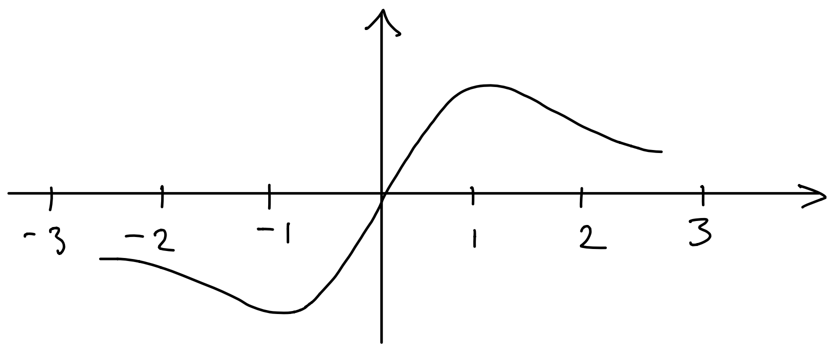
$$\inf \mathbb{N} = 1$$

iv) The set A given by

$$A = \left\{ \frac{2x}{1+x^2} \,\middle|\, x \in \mathbb{R} \right\}$$

given by the graph of the function

$$x \longmapsto \frac{2x}{1+x^2} :$$



We have $(1-x)^2 \geq 0, \ \forall \, x \in \mathbb{R}$

$\Rightarrow \quad 1 - 2x + x^2 \geq 0 \Leftrightarrow 1 + x^2 \geq 2x$

$\Rightarrow \quad \sup A \leq 1$. On the other hand, from setting $x = 1$, we obtain $\sup A \geq 1$

$\Rightarrow \quad \sup A = 1$

Let now $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$.

<u>Definition 3.5:</u>

$(a_n)_{n \in \mathbb{N}}$ is bounded from above (below), if

$$\exists \, b \in \mathbb{R}, \ \forall \, n \in \mathbb{N}: a_n \leq b \quad (b \leq a_n),$$

that is, if the set $A = \{ a_n \mid n \in \mathbb{N} \}$ is bounded from above (below).

Proposition 3.7:

If $(a_n)_{n\in\mathbb{N}}$ is convergent, then $(a_n)_{n\in\mathbb{N}}$ is also bounded.

Proof:

Let $a = \lim_{n\to\infty} a_n$, and for $\varepsilon = 1$ let $n_0 \in \mathbb{N}$ be such that $|a_n - a| < 1$ for $n \geq n_0$

For $n \geq n_0$ we then have

$$|a_n| = |a_n - a + a| \leq |a| + |a_n - a| \leq |a| + 1$$

Therefore,

$$\forall\ n \in \mathbb{N} :\ |a_n| \leq \max\{|a|+1, |a_1|, |a_2|, \ldots, |a_{n_0}|\}$$

$\square$

Boundedness is therefore necessary, but <u>not</u> sufficient for convergence.

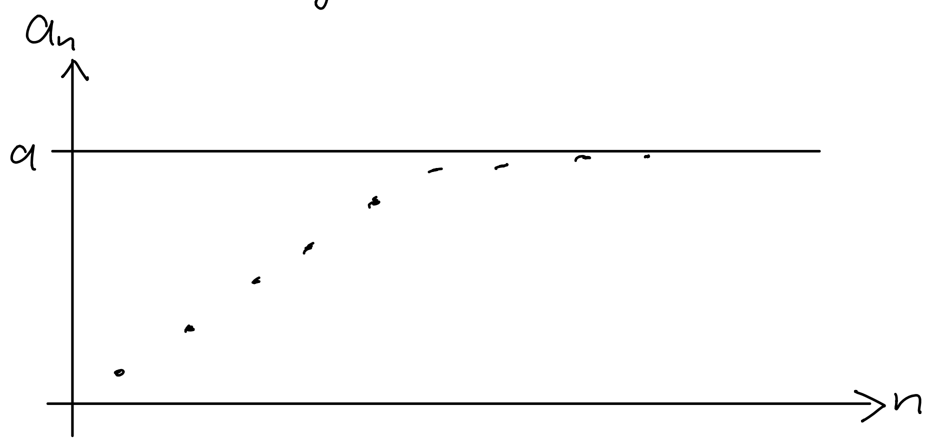
Proposition 3.8:

Let $(a_n)_{n\in\mathbb{N}}$ be bounded from above and "monotonically increasing", that is there exists a number $b \in \mathbb{R}$, s.t.

$$\forall\ n \in \mathbb{N} :\ a_1 \leq a_2 \leq \ldots \leq a_n \leq a_{n+1} \leq \cdots \leq b$$

Then $(a_n)_{n\in\mathbb{N}}$ is convergent, and $\lim_{n\to\infty} = \sup a_n$

Analogously, if $(a_n)_{n\in\mathbb{N}}$ is bounded from below and "monotonically decreasing", then it is convergent.



Proof:

Let $A = \{a_n \mid n \in \mathbb{N}\}$. According to assumption, $A \neq \emptyset$ is bounded from above; therefore there exists $a = \sup A = \sup_{n\in\mathbb{N}} a_n$ according to Prop. 3.6

Claim: We have $a = \lim_{n\to\infty} a_n$

Proof:

Let $\varepsilon > 0$ be arbitrary. Then there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t. $a_{n_0} > a - \varepsilon$. Monotomy then gives

$$\forall\, n \geq n_0 : a - \varepsilon < a_{n_0} \leq a_n \leq \sup_{\ell\in\mathbb{N}} a_\ell = a < a + \varepsilon$$

so that $\forall\, n \geq n_0 : |a_n - a| < \varepsilon$ $\square$

Definition 3.6 (superior/inferior limit):

Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be bounded, that is

$$\exists\, M \in \mathbb{R}\ \forall\, n \in \mathbb{N} : |a_n| < M$$

For $k \in \mathbb{N}$ we then have

$$c_k = \inf_{n\geq k} a_n \leq \sup_{n\geq k} a_n = b_k$$

Aparently, we have

$$-M \leq c_1 \leq \dots \leq c_k \leq c_{k+1} \leq b_{k+1} \leq b_k \leq \dots \leq b_1 \leq M,$$

$$\forall\, k \in \mathbb{N}$$

Prop. 3.8

$$\Rightarrow \exists\ b = \lim_{k\to\infty} b_k =: \lim_{n\to\infty} \sup a_n \text{ ("superior limit")}$$

$$c = \lim_{k\to\infty} c_k =: \lim_{n\to\infty} \inf a_n \text{ ("inferior limit")}$$